Comment on "Time-dependent entropy of simple quantum model systems"

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In the above mentioned paper by J. Dunkel and S. A. Trigger [Phys. Rev. **A 71**, 052102, (2005)] a hypothesis has been pursued that the loss of information associated with the quantum evolution of pure states, quantified in terms of an increase in time of so-called Leipnik's joint entropy, could be a rather general property shared by many quantum systems. This behavior has been confirmed for the unconfined model systems and properly tuned initial data (maximally classical states). We provide two particular examples which indicate a complexity of the quantum evolution. In the presence of a confining (harmonic) potential Leipnik's entropy may be non-increasing for maximally classical initial data. Another choice of initial data implies periodicity in time of the Leipnik entropy.

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The von Neumann entropy is insensitive to the unitary quantum dynamics and in addition vanishes for pure states. Hence, from the modern quantum information-theory point of view pure quantum states (including elaborate text-book discussions of the wave-packet dynamics) appear to be useless. However, there are available other information-theoretic entropy tools which quantify both the "information content" (here, complementarily interpreted as the "uncertainty content") of quantum wave-packets and give account of their Schrödinger picture dynamics.

A straightforward generalization of the Shannon entropy for Born postulate-inferred continuous probability distributions, named the Leipnik entropy, has been used in Refs. [1, 2] to quantify the loss of information associated with temporally evolving pure quantum states. For simple model system examples considered by [1] , initial time $t = 0$ states were chosen to minimize both the Heisenberg uncertainty relation and the joint entropy. They were named MACS (maximal classical states).

For unconfined model systems investigated in [1], it was found that: "A quantum system that has been in a MACS at time $t = 0$ inevitably evolves into a non-MACS at times $t > 0$. This intrinsic property of quantum systems is e.g. reflected by a monotonous increase of the joint entropy." This particular observation has led the Authors to conclude that "most likely, this quantum trend also manifests itself for other types of initial wave packets and external potentials, as well as in many-particle systems".

In the present Comment we wish to indicate that, in the presence of the confining (harmonic) potential, the MACS initial data may preserve the MACS property in time, and thence the Leipnik entropy may be conserved in the course of the quantum evolution. Another initial data choice implies that the Leipnik entropy is periodic in time.

Given an $L^2(R)$-normalized function $\psi(x)$ we denote $(\mathcal{F}\psi)(p)$ its Fourier transform. The corresponding probability densities follow: $\rho(x) = |\psi(x)|^2$ and $\tilde{\rho}(p) = |(\mathcal{F}\psi)(p)|^2$.
We introduce the related position and momentum information (differential) entropies: $\mathcal{S}(\rho) \doteq S_q = -\int \rho(x) \ln \rho(x) dx$ and $\mathcal{S}(\tilde{\rho}) \doteq S_p = -\int \tilde{\rho}(p) \ln \tilde{\rho}(p) dp$, where $\mathcal{S}$ denotes the Shannon entropy for a continuous probability distribution, also named differential entropy.

We assume both entropies to take finite values. Then, there holds the familiar entropic uncertainty relation [4]:

$$S_q + S_p \geq (1 + \ln \pi) \, . \tag{1}$$

Up to an irrelevant additive constant (we shall pass to natural units with $\hbar \equiv 1$), the left-hand-side of the entropic inequality coincides with the Leipnik joint entropy S_J discussed in ref. [1].

In the above, no explicit time-dependence has been indicated, but all derivations go through with any wave-packet solution $\psi(x,t)$ of the Schrödinger equation. The induced dynamics of probability densities may imply the time-evolution of entropies: $S_q(t), S_p(t), S_J(t)$.

If, following conventions we define the squared standard deviation value for an observable A in a pure state ψ as $(\Delta A)^2 = (\psi, [A - \langle A \rangle]^2 \psi)$ with $\langle A \rangle = (\psi, A\psi)$, then for the position X and momentum P operators we have the following version of the entropic uncertainty relation (here expressed through so-called entropy powers, see e.g. [5], $\hbar \equiv 1$):

$$\Delta X \cdot \Delta P \geq \frac{1}{2\pi e} \exp[\mathcal{S}(\rho) + \mathcal{S}(\tilde{\rho})] \geq \frac{1}{2} \tag{2}$$

which is an alternative version of the entropic uncertainty relation.

An important property of the Shannon entropy $\mathcal{S}(\rho)$ is that for any general probability distribution $\rho(x)$ with a fixed variance σ we would have $\mathcal{S}(\rho) \leq \frac{1}{2} \ln(2\pi e \sigma^2)$. $\mathcal{S}(\rho)$ becomes maximized in the set of such densities if and only if ρ is a Gaussian with variance σ. For Gaussian densities $(2\pi e)\Delta X \cdot \Delta P = \exp[\mathcal{S}(\rho) + \mathcal{S}(\tilde{\rho})]$ holds true, but the minimum $1/2$ on the right-hand-side of Eq. (2), is not necessarily reached.

The familiar (Schrödinger's) coherent state of the harmonic oscillator has the property of transforming *all* inequalities of Eq. (2) into an identity, and preserves coherence for all times. Therefore, we have in hands a non-stationary solution of the Schrödinger equation for which S_J is manifestly time-independent, although being

the MACS in the terminology of Ref. [1]. The MACS property is left intact by the Schrödinger time evolution.

Assuming for simplicity $\hbar = m = \omega = 1$ we have:

$$\rho(x,t) = \pi^{-1/2} \exp[-(x - q(t))^2] \tag{3}$$

where $q(t) = q_0 \cos t + p_0 \sin t$. In the present case, [3]:

$$S_q(t) = S_p(t) = (1/2)(1 + \ln \pi)\,. \tag{4}$$

The monotonous increase of the Leipnik entropy is *not* an inevitable consequence of the MACS initial data.

The previously mentioned "quantum trend" as well may not arise for other types of initial wave packets. We shall give a non-MACS example where the indeterminacy relation is saturated, but the Leipnik entropy is not at its extremum.

Let us consider the squeezed wave function of the harmonic oscillator, [3]. We work with the re-scaled units $\hbar = \omega = m = 1$. The solution of the Schrödinger equation $i\partial_t \psi = -(1/2)\Delta\psi + (x^2/2)\psi$ with the initial data $\psi(x,0) = (\gamma^2\pi)^{-1/4} \exp(-x^2/2\gamma^2)$ and $\gamma \in (0,\infty)$, gives rise to the time-dependent probability density :

$$\rho(x,t) = \frac{1}{(2\pi)^{1/2}\sigma(t)} \exp\left(-\frac{x^2}{2\sigma^2(t)}\right) \tag{5}$$

where

$$2\sigma^2(t) = \frac{1}{\gamma^2}\sin^2 t + \gamma^2 \cos^2 t\,. \tag{6}$$

The position entropy reads $S_q = (1/2)\ln[2\pi e \sigma^2(t)]$. The momentum entropy S_p has the same functional form as S_q, except for the replacement of $\sigma^2(t)$ by $\tilde{\sigma}^2(t) = \gamma^2 \sin^2 t + (1/\gamma^2)\cos^2 t$ (that is special to the harmonic oscillator case).

As a consequence, the Leipnik entropy $S_J = S_q + S_p$ is a periodic function of time with a period $\pi/2$ and oscillates between $1 + \ln \pi$ and $1 + \ln \pi + (1/2)\ln[(1/4)(\gamma^4 + \gamma^{-4} + 2)]$, [3]. A similar periodic behavior has been found for prototype Schrödinger cat states, initially modelled as a superposition of two Schrödinger coherent states with the same amplitude but opposite phases.

It is useful to mention that the arithmetic sum of position and momentum entropies has received an ample attention in the literature investigating complex atoms, [6] while the general issue of the Shannon entropy dynamics has been addressed in [7].

Let us add some general comments about the uses of time-dependent entropies in quantum theory.

Among numerous manifestations of the concept of entropy in physics and mathematics, information-theory based entropy methods were devised to investigate the large time behavior of solutions for various (mostly dissipative) partial differential equations. Shannon, Kullback and von Neumann entropies are typical information theory tools which were designed to quantify the "information content" and possibly "information loss" for systems in a specified state.

For quantum systems, the von Neumann entropy vanishes on pure states, hence one presumes to have a "complete information" about the state. On the other hand, for pure states the differential entropy gives access to another "information level", associated with a probability distribution inferred from a $L^2(R^n)$ wave packet. It is perfectly suited to give account of the Schrödinger picture dynamics and this property extends to the Leipnik entropy.

Since, in physical sciences, entropy is typically regarded as a measure of the degree of randomness and the tendency (trends) of physical systems to become less and less "organized", it is quite natural to think of entropy as about the measure of uncertainty. In view of the profound role played by the Shannon entropy in the formulation of entropic indeterminacy relations, the term "information", in the present context should be used in the technical sense, meaning the inverse of "uncertainty".

We may attribute a concrete meaning to the term "organization" in the quantum wave packet context. Namely, Shannon and Leipnik entropies quantify the degree of the probability distribution "complexity", [6], and "(de)localization", [7], for stationary and non-stationary Schrödinger wave packets.

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